Journal of Statistics, Optimization and Data Science
Vol. 1 No: 1 (June 2023); pp 35-49

Combinatorial Probability Distribution for 2ndDiagonal Elements of the Pascal's Triangle with Sequence of Natural Numbers in Prime Diagonal

Madhura A. Gandhi, Tirupathi Rao Padi, Mugdha A. Gandhi

1.Department of Statistics, Savitribai Phule Pune University, India, MS, Email: madhura.gandhi20@gmail.com
2.Professor, Department of Statistics, Pondicherry University, Puducherry, India, Email: drtrpadi@gmail.com
3.Department of Statistics, Savitribai Phule Pune University, India, MS, Email: mugdha.gandhi1@gmail.com

ABSTRACT

The sequence of all positive integers arranged in increasing order is called as the sequence of Natural Numbers. In this paper, we model the probability distribution for the second diagonal elements of Pascal's triangle obtained from the sequence of natural numbers as the first diagonal elements. Explicit functional relation for the p.m.f. of the said distribution has been formulated. Functional forms of different statistical measures have been derived from the developed p.m.f. of the mentioned distribution. This study also derived the other properties like mean, variance, Pearson's coefficient, moment generating functions, characteristic functions etc. In order to understand the model behaviour, numerical illustrations have been applied, from which the statistical measures are obtained from the derived mathematical formulae. The study further expanded by providing the complete statistical analysis with the suitable data visualization tools. This study has good scope and applicability in different areas such as architects, graphic designers, banking, etc.

Keywords: Natural Number's Sequence, Pascal's Triangle, Combinatorial Probability Distribution, Characteristic Function, Pearson's Coefficients.

Corresponding Author: *Madhura A. Gandhi,* madhura.gandhi20@gmail.com

1. Introduction:

1.1. About the Context:

In the recent years, the world is becoming data driven. As the accessibility of data is increasing, data is becoming even more important in our world. Start-ups and businesses across the globe are developing software to collect, store, manage, and analyze data for various strategic purposes like solving problems, decision-making, creating new products, planning marketing strategies, etc. Calculation of the probabilities of events of interest plays a key role for all these purposes. A probability distribution is the mathematical function that gives the chances of occurrence of different possible outcomes for an experiment. It has applications in almost all the areas, finance, health, artificial intelligence, etc. Pascal's triangle is one of the mysterious triangles which is a triangular array of the binomial coefficients that arises in probability theory, Combinatorics and algebra. In simple words, Pascal's triangle is a pattern of numbers in the shape of a triangle, where each number is found by adding the two numbers above it. This paper briefly looks at the modified version of Pascal's triangle, how it is derived and explored the combinatorial probability distribution for 2nd diagonal elements of the derived Pascal's triangle and its characteristics.

1.2. Combinations of Context:

In general Pascal's triangle has all the first diagonal elements are 1. However, in this paper, we propose and discussed the modified version of Pascal's triangle having prime diagonal elements as the sequence of the natural numbers and obtained further elements using the idea of usual Pascal's triangle i.e., by adding numbers of the immediately previous line. After getting Pascal's triangle with prime diagonal as natural numbers, we have discussed combinatorial probability distribution of 2^{nd} diagonal elements of the newly formed Pascal's triangle.

1.3. Significance of the Context:

Combinatorics is especially useful in computer science. Combinatorics methods can be used to develop estimates about how many operations a computer algorithm will require. It can also be used to count possible outcomes in a uniform probability experiment. Pascal's triangle can be used in probability to simplify counting the probabilities of some event. In this paper, we have derived probability mass function, cumulative generating function, Moment generating functions, characteristics of p.m.f. and so on. This study has good scope and applicability in different areas such as Network Designing, Operational Research, Expansion Architecture, Graphics, etc.

1.4. Some Reported Studies:

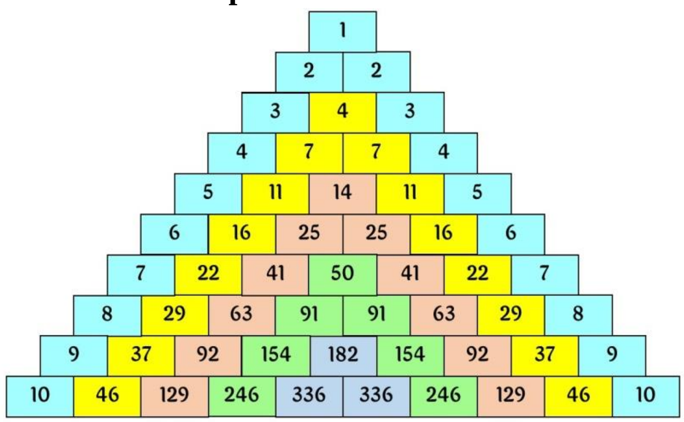
When we try to understand the architecture of Pascal Triangle, the notions of Combinatorics, Arithmetic, and Geometry by Peter Hilton &Jean Pedersen (1987) found that algebraic arguments are more elementary and automatic while combinatorial arguments convey greater insight. Richard C. Bollinger (1993) introduced the concept of extended Pascal triangle, which is the (left justified) array of coefficients in the expansion of $(1 + x + x^2 + \cdots + x^{m-1})^n$ for m, n $\geq$ 0. They have also discussed a few of their properties and applications. The Simple Complexity of Pascal's Triangle by Susan Leavitt (2011) briefly looks at the history of Pascal's triangle and how it was defined and then explores not only its connection with algebra and probability, but also some of the intriguing patterns and topics contained within Pascal's triangle. The approaches to Poisson–Charlier polynomials and probability distribution function by Irem Kucukoglu (2019) have discussed the construction of generating functions for new families of combinatorial numbers and polynomials. By using these generating functions with their functional and differential equations, they not only investigated properties of these new families, but also derived many new identities, relations, derivative formulas, and combinatorial sums.

1.5. Motivation of the Study:

If we want to buy 3 flavours of ice-creams out of 10 flavours then we need to know in how many different ways we can choose those 3. Even this simple situation requires probabilities which can be derived from usual Pascal's triangle. After a thorough search of the literature on Combinatorics probability theory, it is observed that very little work is reported on the formulation of probability distributions and modelling of Combinatorics theory with discrete stochastic processes. Combinatorics has applications in many areas such as computational molecular biology, computer science, machine learning, data science, etc. Pascal's triangle has both theoretical and applied uses. It can be used to derive many formulae in mathematics. The newly derived mathematical formulae will be prospective machine tools in the context of computing aspects and developing the new versions of software. Pascal's triangle is one of the important concepts which can be used for finding probabilities of events of interests. There are many business domains in which calculating chances of some events is required. However, there is no evidence of building probability distribution's models using Combinatorics theory, more specific to the series of natural numbers within different diagonal sequences in Pascal triangle. This study is innovative and interesting. All these factors motivated us to study this topic.

2. Mathematical Modelling

2.1. Schematic Diagram and its Description



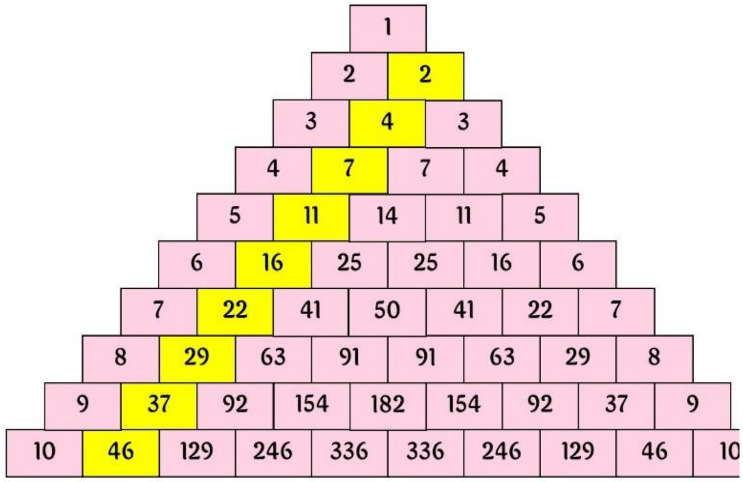
Pascal's Triangle with 1st diagonal elements are sequence of Natural Numbers

Figure-2.1

This is the pyramid of Pascal's triangle with 1st diagonal elements are the sequence of Natural Numbers. Here, we have replaced first diagonal elements by sequence of natural numbers i.e., 1, 2, 3, 4, 5, …, n and similar to the technique of general Pascal's triangle, have calculated next diagonal elements by adding above two numbers. In this paper, we are going to discuss combinatorial probability distribution of 2nd diagonal of the Pascal's triangle. Observing the diagram, it is clearly disclosing that, the First diagonal elements are 1, 2, 3, … n with (n number of elements), Second diagonal elements are 2, 4, 7, … and so on with ($n - 1$ number of elements); Third diagonal elements are 3, 7, 14, … and so on with ($n - 2$ number of elements); on the similar passion the nth diagonal of the triangle will consists of 1 number i.e. 'n'.

2.2. Probability Mass Function of the 2ndDiagonal Element's

Let X be the random variable which denotes the corresponding element in the studied diagonal at i^{th}position of range set of $X's$ say $\{x_1, x_2, \ldots x_i, \ldots, x_{n-1}\}$ which represents elements on 2nd diagonal of Pascal's triangle with prime diagonal elements as sequence of natural numbers.



Pascal's Triangle with 1st diagonal elements are sequence of Natural Numbers

Figure-2.2

2nd diagonal elements are $X = \{2, 4, 7, 11, 16, 22, 29, 37, 46, \ldots\}$. Here, $x_1 = 2$, $x_2 = 4$, $x_3 = 7$, … x_i, …, x_{n-1}. There is total $(n - 1)$ elements in 2nd diagonal. Assume $x_0 = 1$; We can say that, $x_1 = x_0 + 1; x_2 = x_1 + 2; x_3 = x_2 + 3; \ldots\ldots; x_i = x_{i-1} + i; \ldots\ldots; x_{n-1} = x_{n-2} + (n-1)$.

Sum of all the 2nd diagonal elements is

$T = x_1 + x_2 + \cdots + x_i + \cdots + x_{n-1} = \sum_{i=1}^{n-1} x_i$.

Since, $x_i = x_{i-1} + i$ where $x_0 = 1$

$$\text{Therefore, } T = \sum_{i=1}^{n-1}(x_{i-1} + i) = \sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}. \qquad (2.1)$$

Hence, the probability mass function of the combinatorial distribution pertaining to the sequence of 2nd diagonal elements in Pascal's triangle is

$$P(X = x_i) = \begin{cases} \frac{x_i}{\sum_{i=1}^{n-1} x_i}, & i = 1, 2, \ldots, (n-1);\ n > 1; \\ 0\ , & Otherwise \end{cases} \qquad (2.2.)$$

Where; $x_i = x_{i-1} + i \& x_0 = 1$

Another simplified form of the p. m. f. is

$$P(X = x_i) = \begin{cases} \frac{x_{i-1} + i}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}}, & i = 1, 2, \ldots, (n-1);\ n > 1;\ x_0 = 1 \\ 0\ , & Otherwise \end{cases} \qquad (2.3)$$

2.3. Justification / Verification of Derived p. m. f.

A function is a Probability mass function if it satisfies two conditions.

1) $P_i \geq 0; i = 1, 2, \cdots, (n-1)$

Here, $\frac{x_{i-1}+i}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}} \geq 0$ Since all the terms are non-negative. As n > 1; denominator is > 0

2) $\sum_{i=1}^{n-1} P(X = x_i) = \sum_{i=1}^{n-1} \frac{(x_{i-1} + i)}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}} = 1$

Since both the assumptions are satisfied, suggested p. m. f. is verified.

2.4. Cumulative Distribution Function

X being a random variable, its corresponding element at i^{th} position of range set of $X's$ say $\{x_1, x_2, \dots x_i, \dots, x_{n-1}\}$ which represents elements on 2nd diagonal of Pascal's triangle with prime diagonal elements as sequence of natural numbers with probabilities P_i 's. Then cumulative distribution function (c.d.f.) or distribution function (d.f.) is defined as;

$$F(x_i) = P[X \leq x_i] = \sum_{j=1}^{i} P_j, \quad i = 1,2,\dots,(n-1)$$

$$F(x_i) = \sum_{j=1}^{i} \left(\frac{x_{j-1} + j}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}} \right), \quad i = 1,2,\dots,(n-1) \quad (2.4)$$

Since, $X$is the discrete random variable it takes only isolated values. The c.d.f. is constant between two successive values of X and have jumps at the points $x_i, i = 1,2,\dots,(n-1)$. Hence distribution function for our discrete random variable is a step function.

2.5. Statistical Characteristics of the Pascal's 2nd Diagonal Probability Distribution:

2.5.1. Raw moments (Moments about the origin i.e., zero)

The rth order raw moment of random variable $X$is defined as the rth raw moment about 0. It is denoted by μ'_r. This implies $\mu'_r = E(X^r = x_i^r) = \sum_{i=1}^{n-1} x_i^r p_i$; $r = 1, 2, 3, 4, \dots$.

$$\mu'_r = \frac{\sum_{i=1}^{n-1} (x_{i-1} + i)^{r+1}}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}} \quad ; r = 1, 2, 3, 4, \dots \quad (2.5)$$

2.5.1.1. Mean of the distribution:

By putting r = 1 in the above relation, the expected value of X can be obtained.

$$\mu'_1 = \frac{\sum_{i=1}^{n-1} (x_{i-1} + i)^2}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}}$$

After simplification, $Mean = \mu'_1 = \frac{6\sum_{i=1}^{n-1} {x_{i-1}}^2 + 12\sum_{i=1}^{n-1} i x_{i-1} + 2n^3 - 3n^2 + n}{6\sum_{i=1}^{n-1} x_{i-1} + 3n^2 - 3n}$ (2.6)

2.5.1.2. Second Raw Moment:

By substituting r = 2 in the rth order relation, we will get

$$\mu'_2 = \frac{\sum_{i=1}^{n-1} (x_{i-1} + i)^3}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}}$$

On simplification, $\mu'_2 = \frac{4\sum_{i=1}^{n-1} {x_{i-1}}^3 + 12\sum_{i=1}^{n-1} i^2 x_{i-1} + 12\sum_{i=1}^{n-1} i x_{i-1}^2 + n^4 - 2n^3 + n^2}{4\sum_{i=1}^{n-1} x_{i-1} + 2n^2 - 2n}$ (2.7)

2.5.1.3. Third Raw Moment:

Assuming r = 3, $\mu'_3 = \frac{\sum_{i=1}^{n-1} (x_{i-1} + i)^4}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}}$ (2.8)

2.5.1.4. Fourth Raw Moment

Assuming r = 4, $\mu_4' = \frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^5}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}$ (2.9)

2.5.2. Central moments (Moments about the arithmetic mean)

The r^th^ central moment of random variable X is defined as the r^th^ moment of X about $E(X)$. It is denoted by μ_r.

$$\mu_r = \mu_r'\big(E(x)\big) = E[X - E(X)]^r = \sum_{i=1}^{n-1}[x_i - E(X)]^r\, p_i \qquad ; r = 1, 2, 3, 4 \qquad (2.10)$$

Also, we know the relations between Raw moments and Central moments. Using the relations, we can find central moments.

2.5.2.1. Second central moment is given by,

$$\mu_2 = \frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^3}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}} - \left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^2}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)^2 \qquad (2.11)$$

2.5.2.2. Third central moment is given by,

$$\mu_3 = \frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^4}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}} - 3\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^3}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^2}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right) + 2\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^2}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)^3 \qquad (2.12)$$

2.5.2.3. Fourth central moment is given by,

$$\mu_4 = \frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^5}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}} - 4\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^4}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^2}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right) +$$
$$6\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^3}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^2}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)^2 - 3\left(\frac{\sum_{i=1}^{n-1}(x_{i-1}+i)^2}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right)^4 \qquad (2.13)$$

2.5.3. Moment Generating Function

The Moment Generating Function (M.G.F.) of the distribution can be obtained provided$\sum_{i=1}^{n-1} e^{tx_i}p(x_i)$ is convergent for the values of t in some neighbourhood of zero and t is any value belonging to [-a, a], where a is any positive real number.

$$M_X(t) = \sum_{i=1}^{n-1} e^{tx_i}\left(\frac{x_{i-1}+i}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right) = \sum_{i=1}^{n-1}\frac{e^{tx_i}x_{i-1}+e^{tx_i}i}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}} \qquad (2.14)$$

2.5.4. Cumulant Generating Function

The Cumulant Generating Function of a random variable X can be obtained with the MGF as per the regularity conditions of MGF.

$$K_x(t) = log_e\left(\sum_{i=1}^{n-1}\frac{e^{tx_i}x_{i-1}+e^{tx_i}i}{\sum_{i=1}^{n-1}x_{i-1}+\frac{(n-1)(n)}{2}}\right) \qquad (2.15)$$

Journal of Statistics, Optimization and Data Science
Vol. 1 No: 1 (June 2023); pp 35-49

2.5.5. Probability Generating Function

The probability generating function (p.g.f.) of the random variable X is

$$\emptyset(s) = \sum_{i=1}^{n-1} s^{x_i} \left(\frac{x_{i-1}+i}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}} \right) = \frac{\sum_{i=1}^{n-1} s^{x_i}(x_{i-1}+i)}{\sum_{i=1}^{n-1} x_{i-1} + \frac{(n-1)(n)}{2}} \quad (2.16)$$

3. Sensitivity Analysis:

Numerical Illustrations and Discussions were carried out for proper understanding the model behaviour. All the above derived mathematical formulae are considered for numerical illustrations. Computations and data visualization aspects were handled R Software after writing the suitable code of the program. Analytical views on different aspects were arrived with the graphical patterns of the said functions.

3.1. Frequency curves of Probability Mass Function& Cumulative Distribution Function

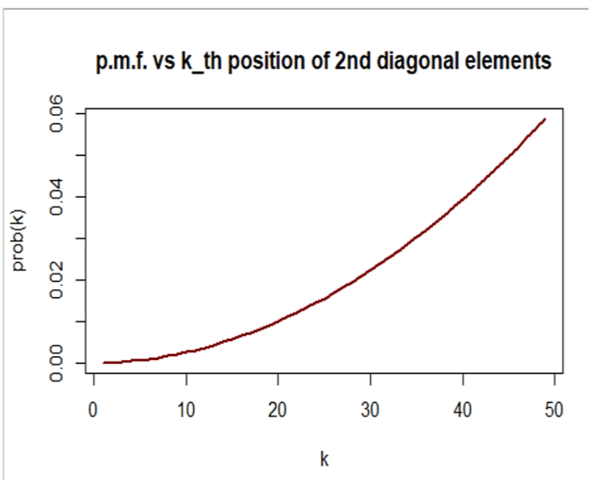


Figure -3.1

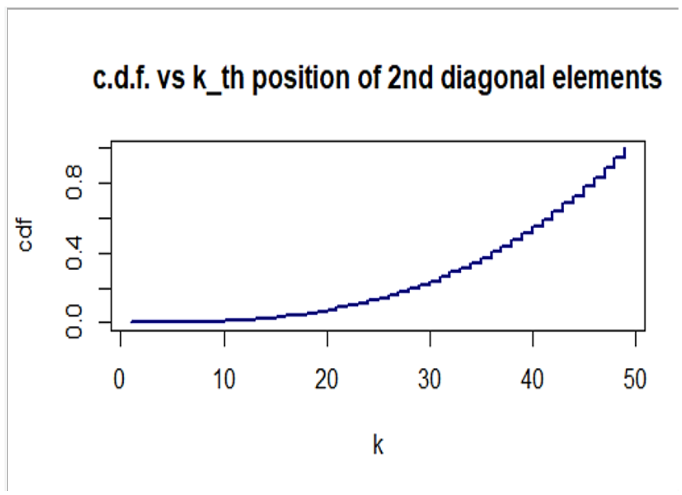


Figure 3.2

As the k i.e., position of the 2nd diagonal element goes on increasing, probability mass function of k also goes on increasing. It is slightly curved, looks like J shaped. Also, derived probability mass function looks like negatively skewed distribution. Further it is resembling the positive relation between k and P (k).

Cumulative Distribution Function is a step function as derived p.m.f. is discrete. Initially values of cumulative distribution function are increasing slowly as k^{th} position of 2nd diagonal increases. But after some value of k, C.D.F. gradually increases and follows J shaped curve.

3.2. Analysis on Statistical Characteristics

3.2.1. The First Four Raw Moments

Graph of 1st, 2nd, 3rd, and 4thraw moments of the probability distribution of 2nddiagonal elements of Pascal's triangle are

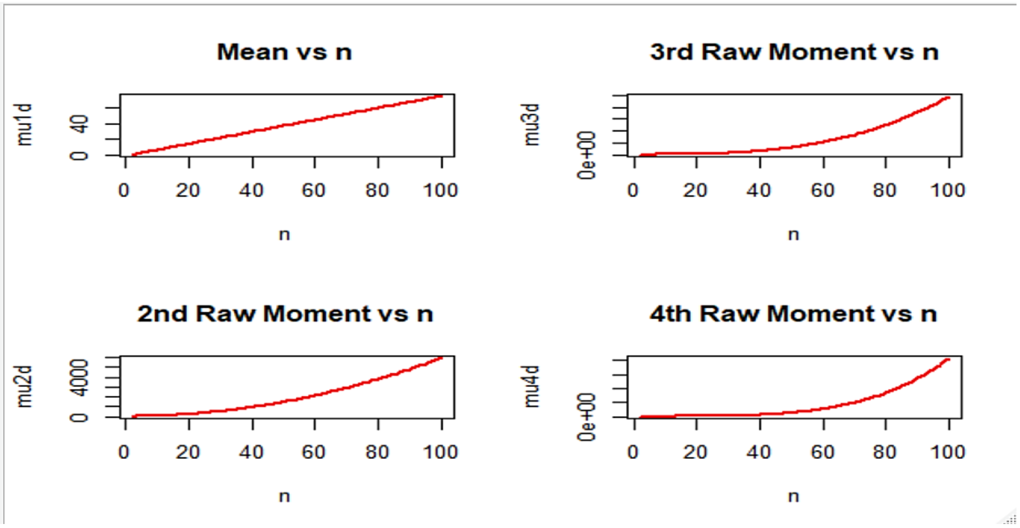
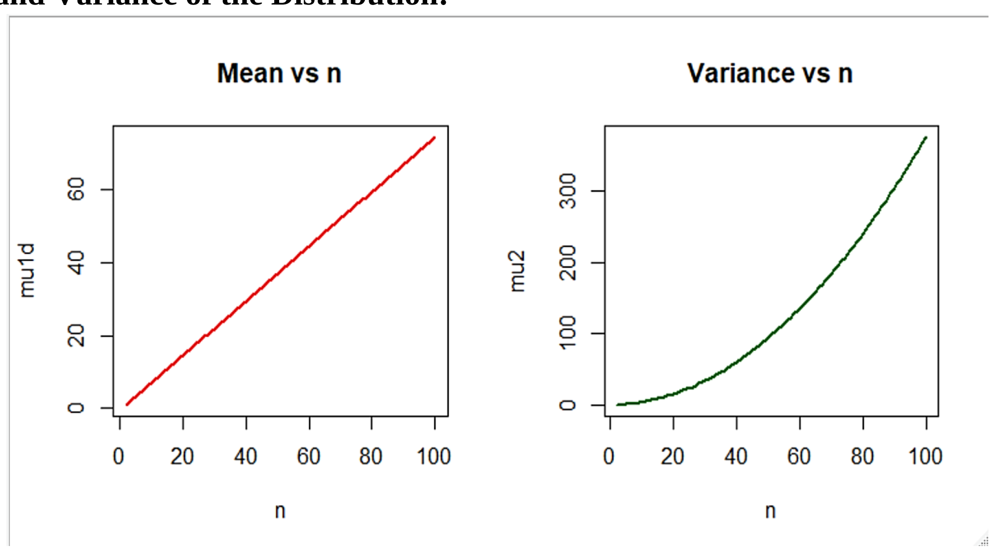


Figure 3.3

As the n i.e., total number of elements in 2^nd^diagonal of Pascal's triangle goes on increasing, the expected value of random variable for each 'n' goes on increasing. It is almost straight line going upward, exhibiting the positive relation between mean and n. The 2nd raw moment of random variable k for each n increases slowly in the beginning. In this graph n = 10, it gradually increases. It is slightly curvilinear. The 3rd raw moment of random variable k for each n is nearly constant or very slowly increasing up to say n = 20 (approximately). But after that it increases gradually. The 4th raw moment of random variable k for each n is nearly constant or very slowly increasing up to say n = 30 (approximately). After that it increases gradually.

The summarized disclosure interprets that as n increases, Graph is straight line for 1st raw moment. For the next three raw moments (2nd, 3rd, 4th…) graph becomes more and more flat in the beginning but after certain value of n, values of raw moment increases gradually and that values are too large for higher order raw moments.

3.2.2. Mean and Variance of the Distribution:



Journal of Statistics, Optimization and Data Science
Vol. 1 No: 1 (June 2023); pp 35-49

Figure 3.4

As the n i.e., total number of elements in 2nd diagonal of Pascal's triangle increases, the mean of random variable will increase linearly for each value of n. On the other hand, for small value of n variance increases slowly in the beginning and after some value of n, variance of k increases gradually. It follows J shaped curve. Also values of variance are comparatively higher than that of mean.

3.2.3. Third and Fourth Central Moments

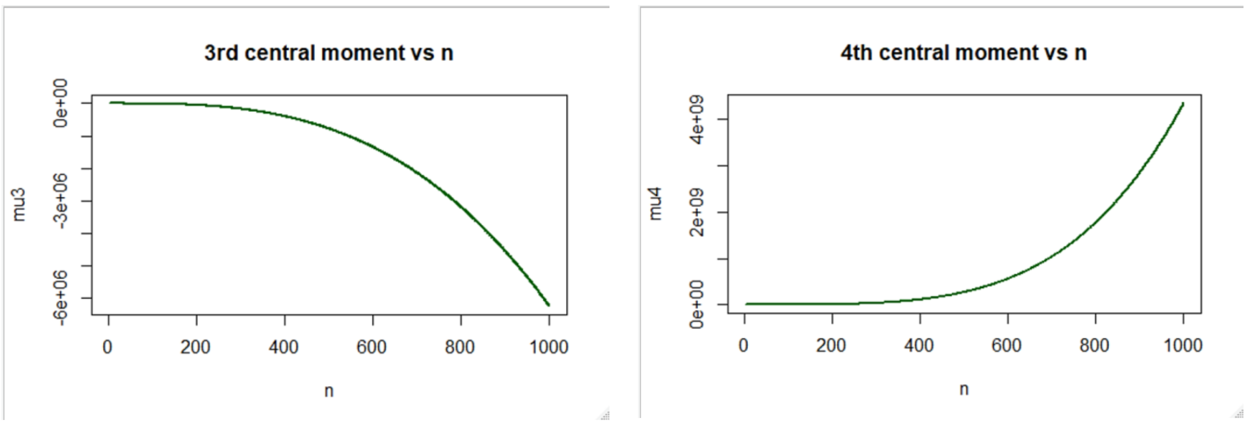


Figure 3.5

For small values of n, μ_3 (3rd central moment) goes on decreasing slowly in the beginning and after some value of n, $\mu_3$of k decreases drastically. Values of 3rd central moment starts from 0 and as n increases, they attain negative values of higher power. For small value of n μ_4 (4th central moment) remains almost constant or may increases very slowly in the beginning and after some value of n, 4th central moment of k increases very fast. It follows J shaped curve and bended in the middle values. Also values of 4th central moments are too higher as n increases. Initially they are 0 and after that increases to very much higher values.

3.2.4. Measures of Skewness (Gamma-1) and Coefficient of Excess (Gamma-2)

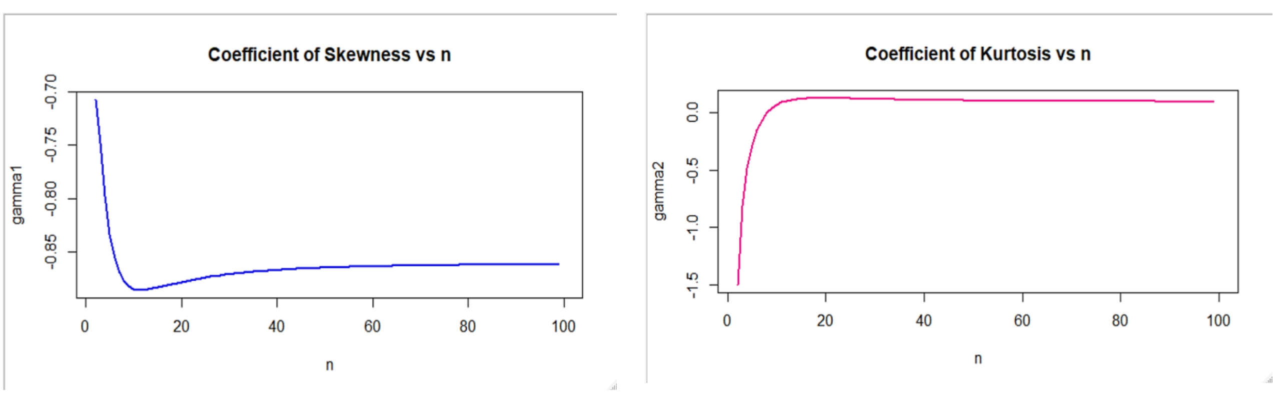


Figure 3.6.

Below are the values of coefficient of skewness calculated for different values of n i.e., total number of elements in 2nd diagonal of Pascal's triangle.

[1] -0.7071068 -0.7479321 -0.7980601 -0.8334670 -0.8561414 -0.8700558
[7] -0.8782889 -0.8829116 -0.8852597 -0.8861842 -0.8862266 -0.8857322
[13] -0.8849217 -0.8839364 -0.8828669 -0.8817704 -0.8806830 -0.8796268
[19] -0.8786146 -0.8776532 -0.8767457 -0.8758925 -0.8750927 -0.8743442
[25] -0.8736445 -0.8729907 -0.8723799 -0.8718091 -0.8712756 -0.8707767
[31] -0.8703097 -0.8698724 -0.8694625 -0.8690779 -0.8687168 -0.8683775
[37] -0.8680583 -0.8677577 -0.8674745 -0.8672073 -0.8669551 -0.8667168
[43] -0.8664914 -0.8662780 -0.8660758 -0.8658841 -0.8657022 -0.8655295
[49] -0.8653653 -0.8652091 -0.8650605 -0.8649189 -0.8647839 -0.8646551
[55] -0.8645322 -0.8644148 -0.8643026 -0.8641953 -0.8640926 -0.8639942

The value of coefficient of skewness is negative since μ_3 (3rd central moment) values are negative. All are negative. At n = 11 (highlighted value) γ_1= -0.8862 which is minimum among all other n. Hence derived probability distribution is Negatively Skewed Distribution.

Below are the values of coefficient of Excess calculated for different values of n i.e., total number of elements in 2nd diagonal of Pascal's triangle.

[1] -1.500000000 -0.808601134 -0.485150026 -0.283764583 -0.150343811
[6] -0.060256509 0.001115176 0.043158750 0.072057850 0.091936136
[11] 0.105570398 0.114848390 0.121065899 0.125120914 0.127641496
[16] 0.129070733 0.129723694 0.129826011 0.129540252 0.128984144
[21] 0.128243270 0.127380007 0.126439867 0.125456031 0.124452624
[26] 0.123447088 0.122451914 0.121475922 0.120525201 0.119603811
[31] 0.118714305 0.117858119 0.117035866 0.116247558 0.115492772
[36] 0.114770775 0.114080621 0.113421218 0.112791387 0.112189897
[41] 0.111615500 0.111066949 0.110543016 0.110042500 0.109564242
[46] 0.109107123 0.108670072 0.108252062 0.107852121 0.107469319
[51] 0.107102778 0.106751663 0.106415188 0.106092605 0.105783210

The values of coefficient of Excess are negative for n i.e., total number of elements in 2nd diagonal of Pascal's triangle equals to 2 to 7. It means that up to 7, probability distribution is platykurtic (less peaked). 8 onwards, coefficient of kurtosis is positive. Hence our probability distribution is leptokurtic (more peaked). For varying values of n, Initially the value of coefficient of kurtosis starts from -1.5. For n equals to 2 to 19, $\gamma_2$increases gradually up to n = 19. At n = 19, coefficient of kurtosis is maximum i.e., 0.1298. After n = 19, $\gamma_2$starts decreasing very slowlyand remains constant around 0.095.

3.3. Special Characteristics

3.3.1. Moment Generating Function

Moment Generating Function is very important function to find the r^{th} order moments of any probability distribution, if exists. Below are the graphs that shows how the values of Moment Generating Function (M.G.F.) varies according the values of n and t. Graph is plotted with the varying values of t between [-a, a], where a is any positive real number. Since for different values of t, the shape of the graph of M.G.F. vs n changes. So, let's summarize the shape of the graph for different values of t, below the graph has 4 different values of t say (0.001, 0.01, 0.1, 1).

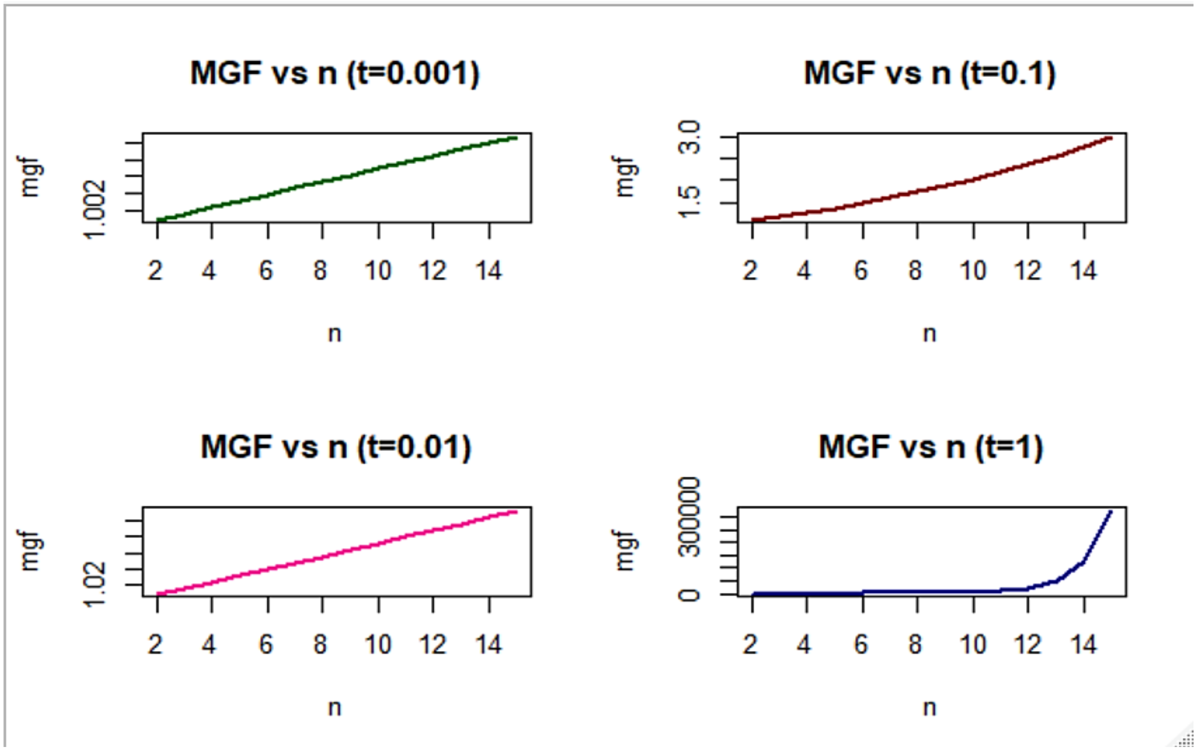


Figure 3.7.

As the value of t increases, values of M.G.F. for large n are too high. Also, the M.G.F. also increases drastically after some value of n. Hence, we should be careful while choosing the value of t. Otherwise t may tend to infinity for large t. For t = 0.001 and 0.01, M.G.F. is increasing in linear pattern. For t = 0.1, graph is slight bended and values are moderate. At t = 1, M.G.F. is too large for large n.

3.3.2. Cumulant Generating Function

Cumulant Generative Function is useful to find central moments easily. Below the graph shows how the values of Cumulant Generating Function (C.G.F.) varies according the n i.e., total number of elements in 1^{st} diagonal of Pascal's triangle. Graph is plotted with the value of t = 0.5.Like M.G.F. Let's check C.G.F. vs n for different values of t (0.001, 0.01, 0.1, 1)

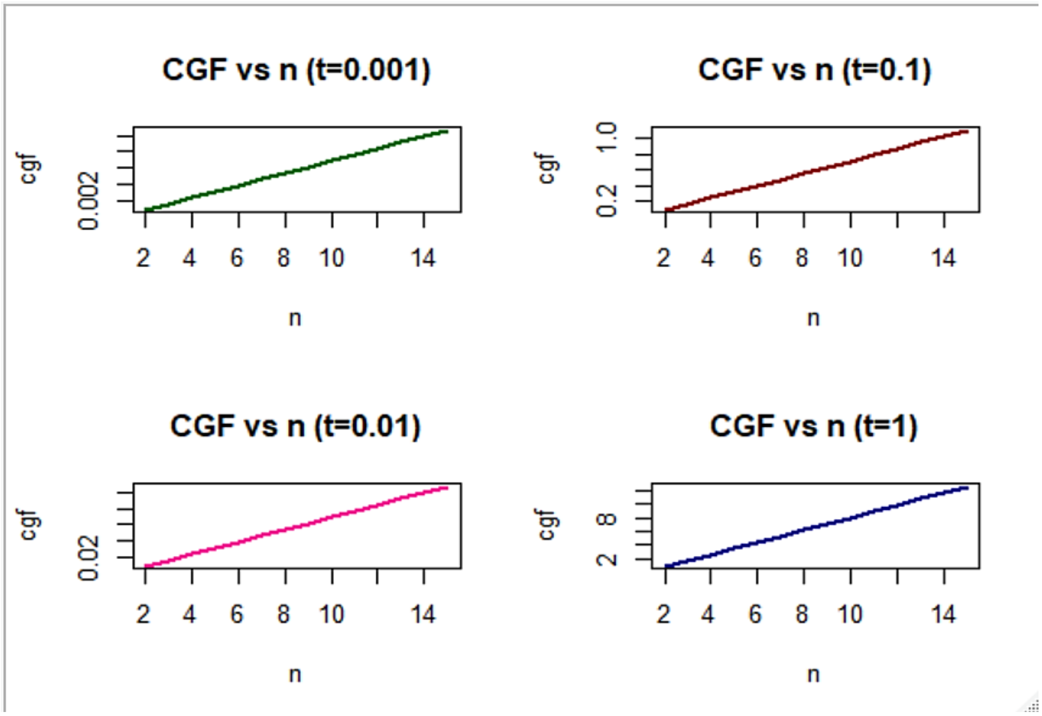


Figure 3.8.

In the case of C.G.F., as the value of t increases, values of C.G.F. are also shifting one decimal place towards right hand side. But for various values of t, C.G.F. verses n follows a straight-line graph. The values of C.G.F. are not too high as compared to t.

3.3.3. Probability Generating Function

A probability generating function (P.G.F.) is a function relating to a probability mass function of a discrete random variable. Below the graphs shows how the values of Probability Generating Function (P.G.F.) varies according to the value of n i.e., total number of elements in 2nddiagonal of Pascal's triangle. Graph is plotted with the value of s = 0.1 where$0 \leq s \leq 1$.

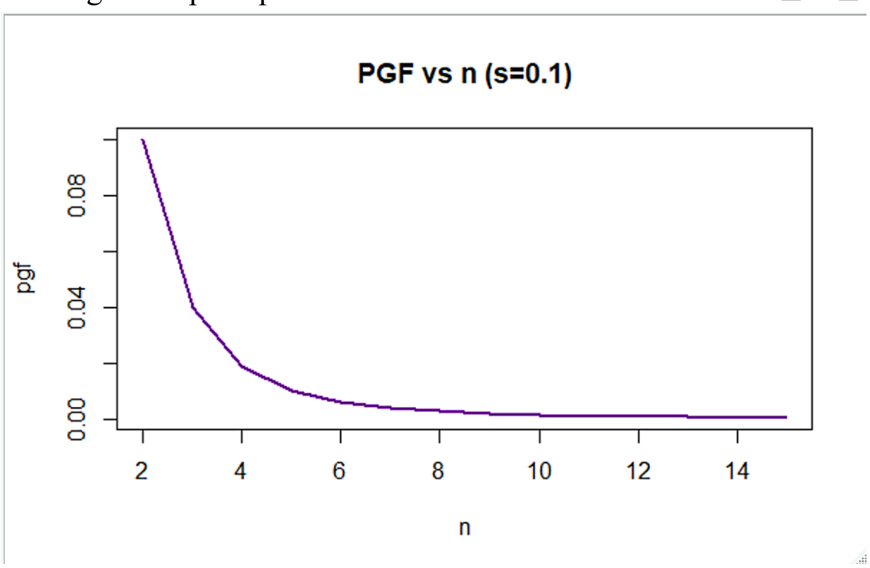


Figure 3.9.

In the case of P.G.F., As the value of n increases, values of P.G.F. are decreasing. First P.G.F. starts from the value of s i.e., here s = 0.1. After that it decreases and for large n P.G.F. is almost tending to 0.

4. Results and Conclusion

4.1. Probability Distribution:

The combinatorial probability distribution of random variable x_i which denotes the elements at i^{th} position on 2nd diagonal of Pascal's triangle with prime diagonal elements as sequence of natural numbers is,

$$P(X = x_i) = \begin{cases} \dfrac{x_{i-1} + i}{\sum_{i=1}^{n-1} x_{i-1} + \dfrac{(n-1)(n)}{2}}, & i = 1, 2, \dots, (n-1);\ n > 1;\ x_0 = 1 \\ 0, & Otherwise \end{cases}$$

As the position of the 2nd diagonal element goes on increasing, probability mass function also goes on increasing. It is slightly curved, looks like J shaped.

4.2 Cumulative Distribution Function:

Initially values of cumulative distribution function are increasing slowly as the position of 2nd diagonal elements increases. But after some value, C.D.F. gradually increases and follows J shaped curve.

4.3. Raw and Central Moments:

As the total number of elements in 1st diagonal of Pascal's triangle (n) increases, mean increases linearly. For the next raw moments (2nd, 3rd, 4th…) graph becomes more and more flat in the beginning but after certain value of n, values of raw moment increases gradually and that values are too large for higher order raw moments. Variance increases gradually with n. Value of 3rd central moment decreases with n. As n increases value of 4th central moment increases little slowly than variance.

4.4. Skewness & Kurtosis:

Probability distribution of an element at i^{th} position on 2nd diagonal of Pascal's triangle with prime diagonal elements as sequence of natural numbers is negatively skewed. Coefficient of Skewness is minimum at n = 11. Probability distribution is Platykurtic up to n = 7 and after that it is Leptokurtic. Coefficient of kurtosis attains its maximum at n = 19.

5. Moment Generating Function:

For a particular value of t, Moment Generating Function increases with n. MGF increases drastically for large values of n. For a particular value of n, Moment Generating Function increases drastically with t.

6. Cumulant Generating Function:

For any value of t, Cumulant Generating Function increases linearly with n.Forany value of n, Cumulant Generating Function increases linearly with t.

7. Probability Generating Function:

PGF decreases with increase in n. For large n, it almost tends to 0.

References:

1. Hilton, P., & Pedersen, J. (1987). Looking into pascal's triangle: Combinatorics, arithmetic, and geometry. Mathematics Magazine, 60(5), 305–316.https://doi.org/10.2307/2690414
2. Kucukoglu, I., Simsek, B., &Simsek, Y. (2019). Generating functions for new families of combinatorial numbers and polynomials: Approach to Poisson–Charlier polynomials and probability distribution function. Axioms, 8(4), 112. https://doi.org/10.3390/axiosdms8040112
3. Richard C. Bollinger (1993) Extended Pascal Triangles, Mathematics Magazine, 66:2, 87-94,https://doi.org/10.1080/0025570X.1993.11996088
4. Kpvs, S. (2015). The Simple Complexity of Pascal's Triangle for the Masters of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of mathematics.https://www.researchgate.net/publication/265269664_The_Simple_Complexity_of_Pascal's_Triangle
5. Balasubramanian, K., Viveros, R., & Balakrishnan, N. Some discrete distributions related to extended pascal triangles.https://www.mathstat.dal.ca/FQ/Scanned/33-5/balasubramanian.pdf
6. W. Feller. (1968) An Introduction to Probability Theory and Its Applications. Vol.1. 3rd ed. New York: Wiley.https://www.academia.edu/41942069/An_Introduction_to_Probability_Theory_and_Its_Applications_Vol_1_3rd_Edition_by_William_Feller_Hardcover
7. Barbarani, V. (2021). Combinatorial models of the distribution of prime numbers. Mathematics, 9(11), 1224.https://doi.org/10.3390/math9111224
8. Chen, C. C., & Koh, K. M. (1992). Principles and techniques in combinatorics. World Scientific.https://idoc.pub/download/principles-and-techniques-in-combinatorics-chen-chuan-chong-koh-khee-meng-ws-1992-6nq8k15r2znw
9. Goldstein, Lay, Schneider. (1996) Calculus and It's Applications. Seventh edition, Prentice-Hall, New York https://www.academia.edu/42803695/Calculus_and_Its_Applications_14th_Edition_by_Larry_J_Goldstein
10. Plaza, Á. (2017). Proof without words: Partial column sums in pascal's triangle. Mathematics Magazine, 90(2), 117–118. https://doi.org/10.4169/math.mag.90.2.117
11. Costello, P. (1990). Analysis of a recursive algorithm for computing binomial coefficients. Computer Science Education, 1(4), 317–329. https://doi.org/10.1080/0899340900010405
12. Hilton, P., Pedersen, J., & Rosenthal, W. (1990). PASCALIAN TRIANGLES AND EXTENSIONS TO HEXAGONS: To our dear friend Keith Hardie on his sixtieth birthday, with

affection and respect. Quaestiones Mathematicae. Journal of the South African Mathematical Society, 13(3–4), 395–416. https://doi.org/10.1080/16073606.1990.9631968